ON METRIZABLE SUBSPACES AND QUOTIENTS OF NON-ARCHIMEDEAN SPACES $C_p(X, \mathbb{K})$

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Abstract. Let $\mathbb{K}$ be a non-trivially valued non-Archimedean complete field. Let $\ell_\infty(\mathbb{N}, \mathbb{K})$ $[\ell_c(\mathbb{N}, \mathbb{K});\ c_0(\mathbb{N}, \mathbb{K})]$ be the space of all sequences in $\mathbb{K}$ that are bounded [relatively compact; convergent to 0] with the topology of pointwise convergence (i.e. with the topology induced from $\mathbb{K}^{\mathbb{N}}$). Let X be an infinite ultraregular space and let $C_p(X, \mathbb{K})$ be the space of all continuous functions from X to $\mathbb{K}$ endowed with the topology of pointwise convergence.

It is easy to see that $C_p(X, \mathbb{K})$ is metrizable if and only if X is countable. We show that for any X [with an infinite compact subset] the space $C_p(X, \mathbb{K})$ has an infinite-dimensional [closed] metrizable subspace isomorphic to $c_0(\mathbb{N}, \mathbb{K})$. Next we prove $C_p(X, \mathbb{K})$ has a quotient isomorphic to $c_0(\mathbb{N}, \mathbb{K})$ if and only if it has a complemented subspace isomorphic to $c_0(\mathbb{N}, \mathbb{K})$. It follows that for any extremally disconnected compact space X the space $C_p(X, \mathbb{K})$ has no quotient isomorphic to the space $c_0(\mathbb{N}, \mathbb{K})$; in particular, for any infinite discrete space D the space $C_p(\beta D, \mathbb{K})$ has no quotient isomorphic $c_0(\mathbb{N}, \mathbb{K})$.

Finally we study the question for which X the space $C_p(X, \mathbb{K})$ has an infinite-dimensional metrizable quotient. We show that for any infinite discrete space D the space $C_p(\beta D, \mathbb{K})$ has an infinite-dimensional metrizable quotient isomorphic to some subspace $\ell_c^0(\mathbb{N}, \mathbb{K})$ of $\mathbb{K}^{\mathbb{N}}$. If $\mathbb{K}$ is locally compact then $\ell_c^0(\mathbb{N}, \mathbb{K}) = \ell_\infty(\mathbb{N}, \mathbb{K})$. If $|n1_{\mathbb{K}}| \neq 1$ for some $n \in \mathbb{N}$, then $\ell_c^0(\mathbb{N}, \mathbb{K}) = \ell_c(\mathbb{N}, \mathbb{K})$. In particular, $C_p(\beta D, \mathbb{Q}_q)$ has a quotient isomorphic to $\ell_\infty(\mathbb{N}, \mathbb{Q}_q)$ and $C_p(\beta D, \mathbb{C}_q)$ has a quotient isomorphic to $\ell_c(\mathbb{N}, \mathbb{C}_q)$ for any prime number q.

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